GENERALIZED N-DERIVATIONS IN PRIME NEAR–RINGS

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ABSTRACT: *The main purpose of this paper is to show that zero symmetric prime left near-rings satisfying certain identities on generalized n- derivation are commutative rings.*

1. INTRODUCTION

A right near – ring (resp. left near ring) is a set N together with two binary operations (+) and (.) such that (i) (N,+) is a group (not necessarily abelian). (ii) (N, .) is a semi group. (iii) For all a,b,c ϵ N ; we have (a + b).c = a.c + b.c (resp. a.(b + c) = a.b + b.c) . Throughout this paper, N will be a zero symmetric left near – ring (i.e., a left near-ring N satisfying the property 0.x = 0 for all x ∈ N). We will denote the product of any two elements x and y in N , i.e.; x.y by xy. The symbol Z will denote the multiplicative centre of N, that is Z = {x ϵ N, xy = yx for all y ϵ N}. A nonempty subset U of N is called semigroup left ideal (resp. semigroup right ideal) if NU ⊆ U (resp. UN ⊆ U) and if U is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. For any x, y ∈ N the symbol [x, y] = xy - yx and (x , y) = x + y – x - y stand for multiplicative commutator and additive commutator of x and y respectively while the symbol x∘y will denote xy + yx. N is called a prime near-ring if xNy = {0} implies that either x = 0 or y = 0. For terminologies concerning near-rings, we refer to Pilz [7].

An additive endomorphism d :N ⟶N is called a derivation if d(xy) = xd(y) + d(x)y, (or equivalently d(xy) = d(x)y + xd(y) for all x, y ϵ N, as noted in [8, proposition 1]. The concept of derivation has been generalized in several ways by various authors. In [3], M. Ashraf defined n-derivations in near-ring and examined some properities of this derivation.

Let n be a fixed positive integer. An n-additive(i.e.; additive in each argument) mapping d:$\underbrace{N\times N\times\ldots\times N}_{n-\text{times}}$ ⟶N is said to be n-derivation if the relations

$d(x_1 x_1', x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)x_1' + x_1 d(x_1', x_2, \ldots, x_n)$

$d(x_1, x_2x_2', \ldots, x_n) = d(x_1, x_2, \ldots, x_n)x_2' + x_2 d(x_1, x_2', \ldots, x_n)$

$d(x_1, x_2, \ldots, x_nx_n') = d(x_1, x_2, \ldots, x_n)x_n' + x_n d(x_1, x_2, \ldots, x_n')$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

In [4], M. Ashraf defined generalized n-derivations in near-ring and studied some properties involved there, which gives a generalization of n-derivation of near-rings.

Let d be n-derivation of a near-ring N. An n-additive mapping f:$\underbrace{N\times N\times\ldots\times N}_{n-\text{times}}$ ⟶N is called a right generalized n-derivation of N with associated n-derivation d if the relations

$f(x_1x_1', x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)x_1' + x_1 d(x_1', x_2, \ldots, x_n)$

$f(x_1, x_2x_2', \ldots, x_n) = f(x_1, x_2, \ldots, x_n)x_2' + x_2 d(x_1, x_2', \ldots, x_n)$

$\vdots$

$f(x_1, x_2, \ldots, x_nx_n') = f(x_1, x_2, \ldots, x_n)x_n' + x_n d(x_1, x_2, \ldots, x_n')$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

An n-additive mapping f:$\underbrace{N\times N\times\ldots\times N}_{n-\text{times}}$ ⟶N is called a left generalized n-derivation of N with associated n-derivation d if the relations

$f(x_1x_1', x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)x_1' + x_1 f(x_1', x_2, \ldots, x_n)$

$f(x_1, x_2x_2', \ldots, x_n) = d(x_1, x_2, \ldots, x_n)x_2' + x_2 f(x_1, x_2', \ldots, x_n)$

$\vdots$

$f(x_1, x_2, \ldots, x_nx_n') = d(x_1, x_2, \ldots, x_n)x_n' + x_n f(x_1, x_2, \ldots, x_n')$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

Lastly an n-additive mapping f: $\underbrace{N\times N\times\ldots\times N}_{n-\text{times}}$ ⟶N is called a generalized n-derivation of N with associated n-derivation d if it is both a right generalized n-derivation as well as a left generalized n-derivation of N with associated n-derivation d.

Many authors studied the relationship between structure of near-ring N and the behaviour of special mapping on N. There are several results in the existing literature which assert that prime near-ring with certain constrained derivations have ring like behaviour. Recently several authors (see [1–6] and [8] for references where further references can be found) have investigated commutativity of near-rings satisfying certain identities. Motivated by these results we shall consider generalized n-derivation on a near-ring N and show that prime near-rings satisfying some identities involving generalized n-derivations and semigroup ideals or ideals are commutative rings.

2. Preliminary Results

We begin with the following lemmas which are essential for developing the proofs of our main results.

Lemma 2.1[6] Let N be a prime near-ring. If z∈ Z \{0} and x is an element of N such that xz ∈ Z or zx∈ Z, then x∈ Z.

Lemma 2.2[6] Let N be a prime near-ring and U a nonzero semigroup ideal of N. If x, y ∈ N and xUy = {0} then x = 0 or y = 0.

Lemma 2.3 [6] Let N be a prime near-ring and Z contains a nonzero semigroup left ideal or nonzero semigroup right ideal, then N is a commutative ring.

Lemma 2.4 [4] Let d be an n-derivation of a near ring N. Then d(Z,N,…,N) ⊆ Z.

Lemma 2.5 [3] Let N be a prime near-ring admitting a nonzero n-derivation d such that d(N, N, . . ., N) ⊆ Z then N is a commutative ring.

Lemma 2.6 [4] Let N be a prime near ring, d a nonzero n-derivation of N, and $U_1, U_2, \ldots, U_n$ be nonzero semigroup left ideals of N. If $d(U_1, U_2, \ldots, U_n) \subseteq Z$, then N is a commutative ring.

Lemma 2.7 [5] Let N be a near-ring. Then f is a left generalized n-derivation of N associated with n-derivation d if and only if

$f(x_1x'_1, x_2, \ldots, x_n) = x_1 f(x'_1, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n)\, x'_1$

$f(x_1,x_2x'_2,\dots,x_n) = x_2f(x_1,x'_2,\dots,x_n) + d(x_1,x_2,\dots,x_n)x'_2$
$f(x_1,x_2,\dots,x_nx'_n) = x_nf(x_1,x_2,\dots,x'_n) + d(x_1,x_2,\dots,x_n)x'_n$
hold for all $x_1, x'_1, x_2, x'_2,\dots,x_n, x'_n \in N$.

Lemma 2.8 [5] Let N be a near-ring admitting a generalized n-derivation f with associated n-derivation d of N. Then

$(d(x_1,x_2,\dots,x_n)x_1' + x_1f(x_1',x_2,\dots,x_n))y = d(x_1,x_2,\dots,x_n)x_1'y + x_1f(x_1',x_2,\dots,x_n)y,$

$(d(x_1,x_2,\dots,x_n)x_2' + x_2f(x_1,x_2',\dots,x_n))y = d(x_1,x_2,\dots,x_n)x_2'y + x_2f(x_1,x_2',\dots,x_n)y,$

$\vdots$

$(d(x_1,x_2,\dots,x_n)x_n' + x_nf(x_1,x_2,\dots,x_n'))y = d(x_1,x_2,\dots,x_n)x_n'y + x_nf(x_1,x_2,\dots,x_n')y,$

for all $x_1, x'_1, x_2, x'_2,\dots, x_n, x'_n, y \in N$.

Lemma 2.9[5] Let N be a near-ring admitting a generalized n-derivation f with associated n-derivation d of N. Then

$(x_1f(x'_1,x_2,\dots,x_n) + d(x_1,x_2,\dots,x_n)x'_1)y = x_1f(x'_1,x_2,\dots,x_n)y + d(x_1,x_2,\dots,x_n)x'_1y,$

$(x_2f(x_1,x'_2,\dots,x_n) + d(x_1,x_2,\dots,x_n)x'_2)y = x_2f(x_1,x'_2,\dots,x_n)y + d(x_1,x_2,\dots,x_n)x'_2y,$

$(x_nf(x_1,x_2,\dots,x'_n)+d(x_1,x_2,\dots,x_n)x'_n)y = x_nf(x_1,x_2,\dots,x'_n)y + d(x_1,x_2,\dots,x_n)x'_n y$

for all $x_1, x'_1, x_2, x'_2,\dots, x_n, x'_n, y \in N$.

Lemma 2.10 [2] Let N be a prime near-ring admitting a left generalized n-derivation f with associated nonzero n-derivation d of N. Let $U_1, U_2, \dots, U_n$ be nonzero semigroup ideals of N. If $f(u_1u'_1, u_2, \dots, u_n) = f(u'_1 u_1, u_2, \dots, u_n)$ for all $u_1, u'_1 \in U_1, u_2 \in U_2,\dots,u_n \in U_n$, then N is a commutative ring.

Lemma 2.11 [2] Let N be a prime near-ring admitting a nonzero generalized n-derivation f with associated n-derivation d of N. Let $U_1, U_2, \dots, U_n$ be nonzero semigroup right ideals of N. If $f(U_1, U_2, \dots, U_n) \subseteq Z$, then N is a commutative ring.

3. MAIN RESULT

Theorem 3.1 Let N be a prime near ring admitting a generalized n-derivation f associated with nonzero n-derivation d of N. Let $U_1,U_2,\dots,U_n$ be semigroup ideals of N. Then the following assertions are equivalent

(i) $f([x, y],u_2,\dots,u_n) = [f(x,u_2,\dots,u_n), y]$ for all $x,y\in U_1,u_2\in U_2,\dots,u_n \in U_n$.

(ii) $[f(x,u_2,\dots,u_n), y] = [x, y]$ for all $x,y\in U_1,u_2\in U_2,\dots,u_n \in U_n$.

(iii) N is a commutative ring.

Proof. It is easy to verify that (iii) ⇒(i) and (iii) ⇒ (ii)

(i) ⇒ (iii) Assume that

$f([x, y],u_2,\dots,u_n) = [f(x,u_2,\dots,u_n),y]$ for all $x, y\in U_1,u_2\in U_2,\dots,u_n \in U_n$. (1)

If we take y = x in (1) we get

$f(x,u_2,\dots,u_n)x = xf(x,u_2,\dots,u_n)$ for all $x\in U_1 ,u_2\in U_2,\dots,u_n\in U_n$. (2)

Replacing y by xy in (1) to get

$f([x, xy],u_2,\dots,u_n) = [f(x,u_2,\dots,u_n), xy]$ for all $x\in U_1 ,u_2\in U_2,\dots,u_n\in U_n$.

Therefore

$f(x[x, y],u_2,\dots,u_n) = [f(x,u_2,\dots,u_n), xy]$ for all $x\in U_1 ,u_2\in U_2,\dots,u_n\in U_n$.

Hence, we get

$d(x,u_2,\dots,u_n)[x, y] + xf([x, y],u_2,\dots,u_n) = [f(x,u_2,\dots,u_n), xy]$ for all $x\in U_1 ,u_2\in U_2,\dots,u_n\in U_n$.

Using (1) again, previous equation implies that

$d(x,u_2,\dots,u_n)[x, y] + x[f(x,u_2,\dots,u_n), y] = [f(x,u_2,\dots,u_n), xy]$ for all $x\in U_1,u_2\in U_2,\dots,u_n\in U_n$.

Which means that

$d(x,u_2,\dots,u_n)[x,y] + xf(x,u_2,\dots,u_n)y - xyf(x,u_2,\dots,u_n) = f(x,u_2,\dots,u_n) xy - xyf(x,u_2,\dots,u_n)$ for all $x\in U_1, u_2\in U_2,\dots,u_n\in U_n$.

Using (2) previous equation can be reduced to

$d(x,u_2,\dots,u_n)xy = d(x,u_2,\dots,u_n)yx$ for all $x, y\in U_1, u_2\in U_2,\dots,u_n \in U_n$.

(3)

Replacing y by yr, where $r\in N$, in (3) and using it again to get

$d(x,u_2,\dots,u_n)U_1[x, r] = 0$ for all $x\in U_1, u_2\in U_2,\dots,u_n\in U_n, r\in N$.

(4)

Using Lemma 2.2 in (4), we conclude that

for each $x\in U_1$ either $x\in Z$ or $d(x,u_2,\dots,u_n) = 0$ for all $u_2\in U_2,\dots,u_n \in U_n$, but using Lemma 2.4 lastly, we get $d(x,u_2,\dots,u_n)\in Z$ for all $x\in U_1,u_2\in U_2,\dots,u_n\in U_n$. So we get $d(U_1,U_2,\dots,U_n) \subseteq Z$. Now by using Lemma 2.6 we find that N is a commutative ring.

(ii) ⇒ (iii) suppose that

$[f(x,u_2,\dots,u_n),y] = [x, y]$ for all $x, y\in U_1,u_2\in U_2,\dots,u_n \in U_n$.

(5)

If we take y = x in (5), we get

$f(x,u_2,\dots,u_n)x = xf(x,u_2,\dots,u_n)$ for all $x\in U_1, u_2\in U_2,\dots,u_n \in U_n$.

(6)

Replacing x by yx in (5) and using it again, we get

$[f(yx,u_2,\dots,u_n),y] = [yx, y] = y[x, y] = y[f(x,u_2,\dots,u_n), y]$ for all $x, y\in U_1,u_2\in U_2,\dots,u_n\in U_n$.

So we have

$f(yx,u_2,\dots,u_n)y - yf(yx,u_2,\dots,u_n) = yf(x,u_2,\dots,u_n)y - y^2f(x,u_2,\dots,u_n)$

for all $x,y\in U_1,u_2\in U_2,\dots,u_n\in U_n$.

In view of Lemmas 2.7 and 2.9 the last equation can be rewritten as

$yf(x,u_2,\dots,u_n)y + d(y,u_2,\dots,u_n)xy - (y^2f(x,u_2,\dots,u_n) + yd(y,u_2,\dots,u_n)x) = yf(x,u_2,\dots,u_n)y - y^2f(x,u_2,\dots,u_n)$ for all $x,y\in U_1,u_2\in U_2,\dots,u_n\in U_n$.

So we have

$d(y,u_2,\dots,u_n)xy = yd(y,u_2,\dots,u_n)x$ for all $x, y\in U_1,u_2\in U_2,\dots,u_n\in U_n$.

(7)

Replacing x by xr in (7) and using it again to get

$d(y,u_2,\dots,u_n)xry = yd(y,u_2,\dots,u_n)xr = d(y,u_2,\dots,u_n)xyr$.

Therefore

$d(y,u_2,\dots,u_n)U_1[y,r] = 0$ for all $y\in U_1 ,u_2\in U_2,\dots,u_n\in U_n, r\in N$.

(8)

Since equation (8) is the same as equation (4), arguing as in the proof of (i) ⇒ (iii) we find that N is a commutative ring.

Corollary 3.2 Let N be a prime near ring admitting a generalized n-derivation f associated with nonzero n-derivation d of N. Then the following assertions are equivalent

(i) $f([x_1, y],x_2,\dots,x_n) = [f(x_1,x_2,\dots,x_n), y]$ for all $x_1,x_2,\dots,x_n, y \in N$.

(ii) $[f(x_1,x_2,...,x_n), y] = [x_1, y]$ for all $x_1,x_2,..., x_n ,y \in N$.

(iii) N is a commutative ring.

Corollary 3.3 [1, Theorem 2.1] Let N be a prime near ring which admits a nonzero n-derivation d, if $U_1,U_2, ...,U_n$ are semigroup ideals of N, then the following assertions are equivalent

(i) $d([x, y],u_2,...,u_n) = [d(x,u_2,...,u_n), y]$ for all $x, y\in U_1$, $u_2\in U_2,...,u_n\in U_n$.

(ii) $[d(x,u_2,...,u_n), y] = [x, y]$ for all $x, y\in U_1,u_2\in U_2,...,u_n\in U_n$.

(iii) N is a commutative ring.

Theorem 3.4 Let N be prime near ring admitting a nonzero right generalized n-derivation f associated with n-derivation d of N. If $U_1,U_2,...,U_n$ are nonzero semigroup ideals of N. Then the following assertions are equivalent

(i) $[f(u_1,u_2,...,u_n),y]\in Z$ for all $u_1\in U_1,u_2\in U_2,...,u_n \in U_n,y\in N$.

(ii) N is a commutative ring.

Proof. It is clear that (ii) ⇒ (i)

(i) ⇒ (ii) Suppose that

$[f(u_1,u_2,...,u_n),y] \in Z$ for all $u_1\in U_1,u_2\in U_2,...,u_n \in U_n$, $y\in N$. (9)

Replacing y by $f(u_1,u_2,...,u_n)y$ in (9) to get

$[f(u_1,u_2,...,u_n), f(u_1,u_2,...,u_n)y] \in Z$ for all $u_1\in U_1 ,u_2\in U_2,...,u_n \in U_n$, $y\in N$.

Which means that

$[[f(u_1,u_2,...,u_n), f(u_1,u_2,...,u_n)y],t] = 0$ for all $u_1\in U_1,u_2\in U_2,...,u_n \in U_n$ and $y, t\in N$.

Therefore, we get

$[f(u_1,u_2,...,u_n) [f(u_1,u_2,...,u_n),y],t] = 0$ for all $u_1\in U_1 ,u_2\in U_2,...,u_n \in U_n$, $y,t\in N$.

Hence,

$f(u_1,u_2,...,u_n) [f(u_1,u_2,...,u_n),y]t = tf(u_1,u_2,...,u_n) [f(u_1,u_2,...,u_n),y]$

for all $u_1\in U_1 ,u_2\in U_2,...,u_n \in U_n$, $y,t\in N$.

Using (9) in previous equation implies that

$[f(u_1,u_2,...,u_n),y][f(u_1,u_2,...,u_n),t] = 0$

for all $u_1\in U_1 ,u_2\in U_2,...,u_n \in U_n$ and $y, t \in N$. (10)

In view of (9), equation (10) assures that

$[f(u_1,u_2,...,u_n),y]N[f(u_1,u_2,...,u_n) ,y] = 0$ for all $u_1\in U_1 ,u_2\in U_2,...,u_n \in U_n ,y\in N$.

Primeness of N shows that $[f(u_1,u_2,...,u_n),y] = 0$ for all $u_1\in U_1 ,u_2\in U_2,...,u_n \in U_n$, $y \in N$.

Hence $f(U_1,U_2,...,U_n)\subseteq Z$. By Lemma 2.11 we conclude that N is a commutative ring.

Corollary 3.5 Let N be prime near ring admitting a nonzero right generalized n-derivation f associated with n-derivation d of N. Then the following assertions are equivalent

(i) $[f(x_1,x_2,...,x_n),y] \in Z$ for all $x_1 ,x_2,...,x_n ,y \in N$.

(ii) N is a commutative ring.

Corollary 3.6 [1, Theorem 2.9] Let N be a prime near ring admitting a nonzero n-derivation d of N. If $U_1,U_2,...,U_n$ are nonzero semigroup ideals of N. Then the following assertions are equivalent

(i) $[d(u_1,u_2,...,u_n),y] \in Z$ for all $u_1\in U_1$, $u_2\in U_2,...,u_n\in U_n ,y\in N$.

(ii) N is a commutative ring.

Theorem 3.7 Let N be a 2-torsion free prime near ring, then there exists no generalized n-derivation f associated with nonzero n-derivation d of N such that

$f(x_1,x_2,...,x_n)\circ y = x_1\circ y$ for all $x_1 ,x_2,...,x_n ,y\in N$.

Proof. Suppose that

$f(x_1,x_2,...,x_n)\circ y = x_1\circ y$ for all $x_1 ,x_2,...,x_n$, $y\in N$. (11)

Replacing x_1 by yx_1 in (15) and using it again, we get

$f(yx_1,x_2,...,x_n)\circ y = (yx_1)\circ y$

$= y(x_1\circ y)$

$= y(f(x_1,x_2,...,x_n)\circ y)$

Using Lemma 2.7 and Lemma 2.9 in previous equation, we obtain

$yf(x_1,x_2,...,x_n)y + d(y,x_2,...,x_n)x_1y + yd(y,x_2,...,x_n)x_1+ y^2f(x_1,x_2,...,x_n) =$

$yf(x_1,x_2,...,x_n)y + y^2f(x_1,x_2,...,x_n)$ for all $x_1,x_2,...,x_n,y\in N$.

Hence we get

$d(y,x_2,...,x_n)x_1y = - yd(y,x_2,...,x_n)x_1$ for all $x_1,x_2,...,x_n,y\in N$. (12)

Replacing x_1 by zx_1 in (12),where $z\in N$, we get

$d(y,x_2,...,x_n)zx_1y = - yd(y,x_2,...,x_n)zx_1$

$= (yd(y,x_2,...,x_n)z)(- x_1)$

$= d(y,x_2,...,x_n)z(- y)(- x_1)$ for all $x_1,x_2,...,x_n ,y, z\in N$.

Since $- d(y,x_2,...,x_n)zx_1y = d(y,x_2,...,x_n)zx_1(-y)$ for all $x_1 ,x_2,...,x_n ,y, z\in N$.

The last expression reduced to

$d(y,x_2,...,x_n)zx_1(- y) = d(y,x_2,...,x_n)z(- y)x_1$ for all $x_1, x_2,...,x_n, y, z\in N$.

Taking - y instead of y in previous equation, we get

$d(-y,x_2,...,x_n)zx_1y = d(-y,x_2,...,x_n)zyx_1$ for all $x_1 ,x_2,...,x_n ,y, z\in N$.

So that $d(-y,x_2,...,x_n)z[x_1, y] = 0$ for all $x_1 ,x_2,...,x_n ,y, z\in N$.

Therefore, $d(-y,x_2,...,x_n)N[x_1, y] = \{0\}$ for all $x_1,x_2,...,x_n ,y\in N$.

Primeness of N yields that for each $y\in N$,

either $d(y,x_2,...,x_n) = - d(-y,x_2,...,x_n) = 0$ for all $x_2,...,x_n\in N$ or $y\in Z$.

Using Lemma 2.4 lastly, we get $d(y,x_2,...,x_n)\in Z$ for all $y,x_2,...,x_n\in N$. Hence we conclude that $d(N,N,...,N)\subseteq Z$ and using Lemma 2.5 implies that N is a commutative ring. Since N is 2-torsion free, therefore (11) assures that

$f(x_1,x_2,...,x_n)y = x_1y$ for all $x_1 ,x_2,...,x_n,y\in N$. (13)

Replacing x_1 by x_1t in (13) and using it again, we get $d(x_1,x_2,...,x_n)ty = 0$ for all $x_1, x_2,...,x_n,y,t\in N$. Therefore $d(x_1,x_2,...,x_n)Ny = 0$ for all $x_1,x_2,...,x_n,y\in N$. Primeness of N implies that either d = 0 or y = 0 for all $y\in N$; a contradiction.

Corollary 3.8 [1, Theorem 2.13] Let N be a 2-torsion free prime near ring, then there exists no nonzero n-derivation d of N such that $d(x_1,x_2,...,x_n)\circ y = x_1\circ y$ for all $x_1,x_2,..., x_n,y\in N$.

Theorem 3.9 Let N be 2-torsion free a prime near ring which admits a nonzero right generalized n-derivation f associated with n-derivation d. If $f(x_1,x_2,...,x_n)\circ y \in Z$ for all $x_1 ,x_2,...,x_n ,y \in N$, then N is a commutative ring.

Proof. By our hypothesis, we have

$f(x_1,x_2,...,x_n)\circ y \in Z$ for all $x_1 , x_2,...,x_n ,y\in N$. (14)

(a) If $Z = \{0\}$, then equation (14) reduced to

$yf(x_1,x_2,...,x_n) = - f(x_1,x_2,...,x_n)y$ for all $x_1,x_2,...,x_n$, $y\in N$. (15)

Replacing y by ry, where $r\in N$, in (15) to get

$ryf(x_1,x_2,...,x_n) = - f(x_1,x_2,...,x_n)ry$

$= f(x_1,x_2,...,x_n)r(- y)$

$= rf(-x_1,x_2,...,x_n)(-y)$ for all $x_1,x_2,...,x_n ,y,r \in N$.

Thus we get

$r(yf(x_1,x_2,...,x_n) + f(-x_1,x_2,...,x_n) y) = 0$ for all $x_1,x_2,...,x_n ,y,r \in N$.

Replacing x_1 by $-x_1$ in last eqation we get

$r(-yf(x_1,x_2,...,x_n) + f(x_1,x_2,...,x_n)\, y) = 0$ for all $x_1,x_2,...,x_n ,y,r \in N$.

which implies that

$rN(-yf(x_1,x_2,...,x_n) + f(x_1,x_2,...,x_n)y) = \{0\}$ for all $x_1,x_2,...,x_n,y,r \in N$.

Primeness of N implies that $f(N,N,..,N) \subseteq Z$ and thus $f = 0$, which contradicts our hypothesis, consequently, there exists an element $z \in Z$ such that $z \neq 0$

$f(x_1,x_2,...,x_n)\circ y \in Z$ for all $x_1 ,x_2,...,x_n ,y \in N$. Then $f(x_1,x_2,...,x_n)\circ z = f(x_1,x_2,...,x_n)z + zf(x_1,x_2,...,x_n) \in Z$ for all $x_1,x_2,...,x_n,y \in N$, $z \in Z$. which implies that

$z(f(x_1,x_2,...,x_n) + f(x_1,x_2,...,x_n)) \in Z$, by Lemma 2.1 we find that

$f(x_1,x_2,...,x_n) + f(x_1,x_2,...,x_n) \in Z$ for all $x_1,x_2,...,x_n \in N$. (16)

Moreover from (14) it follows that

$f(x_1+ x_1,x_2,...,x_n)\circ y \in Z$ for all $x_1 ,x_2,...,x_n ,y \in N$.

Which means that

$f(x_1+ x_1,x_2,...,x_n)y + yf(x_1+ x_1,x_2,...,x_n) = (f(x_1,x_2,...,x_n) + f(x_1,x_2,...,x_n))y + y(f(x_1,x_2,...,x_n) + f(x_1,x_2,...,x_n)) \in Z$ for all $x_1 ,x_2,...,x_n ,y \in N$.

Which because of (16), yields that

$(f(x_1 + x_1,x_2,...,x_n) + f(x_1 + x_1,x_2,...,x_n))y \in Z$ for all $x_1 ,x_2,...,x_n, y \in N$.

Therefore, for all $x_1 ,x_2,...,x_n, y, t \in N$ we have

$$(f(x_1 + x_1,x_2,...,x_n) + f(x_1 + x_1,x_2,...,x_n))ty = y(f(x_1 + x_1,x_2,...,x_n) + f(x_1 + x_1,x_2,...,x_n))t = (f(x_1 + x_1,x_2,...,x_n) + f(x_1 + x_1,x_2,...,x_n))yt$$

So that

$(f(x_1 + x_1,x_2,...,x_n) + f(x_1 + x_1,x_2,...,x_n))N[t, y] = \{0\}$ for all $x_1 ,x_2,...,x_n, y, t \in N$.

In view of the primeness and 2-torsion freeness of N, the previous equation yields either $f(x_1 + x_1,x_2,...,x_n) + f(x_1 + x_1,x_2,...,x_n) = 0$ and thus $f = 0$, a contradiction, or $N \subseteq Z$ and N is a commutative ring by Lemma 2.3.

Corollary 3.10 [1, Theorem 2.16] Let N be 2-torsion free a prime near ring which admits a nonzero n-derivation d. If $d(x_1,x_2,...,x_n)\circ y \in Z$ for all $x_1,x_2,...,x_n,y \in N$, then N is a commutative ring.

The following example proves that the hypothesis of primness in various theorems is not superfluous.

Let S be a 2-torsion free commutative near-ring. Let us define :

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x, y, 0 \in S \right\}$$ is zero symmetric near-ring with regard to matrix addition and matrix multiplication. Define $d: \underbrace{N \times N \times \ldots \times N}_{n-times} \longrightarrow N$ such that

$$f\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1x_2 \ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & y_1y_2 \ldots y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to verify that f is generalized n-derivation associated with a nonzero n-derivation d of N satisfying the following conditions for all $A,B,A_1,A_2,...,A_n \in N$,

(i) $f([A,B],A_2,...,A_n) = [f(A,A_2,...,A_n),B]$

(ii) $[f(A,A_2,...,A_n),B] = [A,B]$

(iii) $[f(A_1,A_2,...,A_n),B] \in Z$ for all $A_1 ,A_2,...,A_n ,B \in N$.

(iv) $f(A_1,A_2,...,A_n)\circ B = A_1\circ B$

(v) $f(A_1,A_2,...,A_n)\circ B \in Z$

However, N is not a ring.

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